Descriptive Set Theory HW 5

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Problem 1. Let X be Polish and let $\{A_n\}_{n\in\omega}$ be a sequence of disjoint analytic sets in X. Prove that there are disjoint Borel sets $\{B_n\}_{n\in\omega}$ with $B_n \supseteq A_n$.

Solution. First, we state the following claim, which follows from using Luzin's analytic separation theorem pairwise and then taking a countable intersection of all the witnesses to this result:

Claim 1. Given a countable collection $\{A_n\}_{n\in\omega}$ of analytic sets and a fixed $i < \omega$, there's a Borel set $B \supseteq A_i$ such that $A_j \subseteq B^c$ for each $j \neq i$.

Then, we construct the $B_i's$ by recursion where at the n^{th} stage we use the above claim to find a B_n that separates A_n from all A_m for $m > n$ and all B_k for $k < n$. This implies that the collection $\{B_n\}_{n\in\omega}$ is pairwise disjoint because if $n < m$ then $B_n \subseteq B_m^c$ by construction. The result follows. ⋆

Problem 2. Let X be Polish and let E be an analytic equivalence relation on X.

1. Show that for an analytic set A, its saturation $[A]_E$ is also analytic.

2. Let $A, B \subseteq X$ be disjoint invariant analytic sets. Prove that there is an invariant Borel set D separating A and B, i.e. $D \supseteq A$ and $D \cap B = \varnothing$.

Solution.

1. Observe that $x \in [A]_E \Leftrightarrow (\exists y \in A)(x, y) \in E \Leftrightarrow x \in \text{proj}_1((X \times A) \cap E)$. Since $(X \times A) \cap E$ is analytic and analytic sets are closed under continuous images, it follows that $[A]_E$ is analytic as well.

2. Assume A and B are disjoint and invariant analytic sets. We construct an increasing sequence $\{D_n\}_{n<\omega}$ of Borel sets such that $[D_n] \cap B = \varnothing$ for each n as follows: let D_0 be a Borel set separating A and B. Since $D_0 \cap B$ is empty, it follows that $[D_0] \cap B$ is empty because B is invariant. Given D_n, observe that $[D_n]$ is analytic by the previous part of this

problem, and is disjoint from B by the induction hypothesis. Then, let $D_{n+1} \supseteq [D_n]$ be a Borel set separating $[D_n]$ and B. By the same argument as above, $[D_{n+1}] \cap B$ is empty.

So, we have $A \subseteq D_0 \subseteq [D_0] \subseteq \ldots \subseteq D_n \subseteq [D_n] \subseteq \ldots$. Finally, define $D = \bigcup_i D_i$. This is Borel because it's a countable union, it's invariant because $D = \bigcup_i [D_i]$, and it separates A and B by construction.

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Problem 3. Construct an example of a closed equivalence relation E on a Polish space X and a closed set $C \subseteq X$ such that the saturation $[C]_E$ is analytic but not Borel.

Solution. Fix an analytic $A \subseteq \mathcal{N}$ that's analytic but not Borel and let $C \subseteq \mathcal{N}^2$ be a closed set such that $A = \text{proj}_1(C)$. Define $(a,b)E(c,d) \Leftrightarrow a = c$, which is certainly closed. We want to identify points that project to the same thing. Now, $[C]_E = \text{proj}_1((\mathcal{N}^2 \times C) \cap E) \subseteq \mathcal{N}^2$ is analytic like the previous problem. If it were also Borel, consider the continuous function $f \colon x \mapsto (x, x)$ from $\mathcal{N}$ to $\mathcal{N}^2$. Then $f^{-1}([C]_E)$ would also be Borel. But, it's not hard to check that $f^{-1}([C]_E) = A$, contradicting the choice of A. ⋆

Problem 4. Let X be set and let τ_0 , τ_1 be Polish topologies on X such that $\tau_0 \subseteq \mathcal{B}(\tau_1)$. Show that $\mathcal{B}(\tau_0) = \mathcal{B}(\tau_1)$

Solution. Obviously $\mathcal{B}(\tau_0) \subseteq \mathcal{B}(\tau_1)$. To show the other direction, consider the identity map $i \colon (X, \tau_1) \to (X, \tau_0)$. The hypothesis implies this is a Borel map. By the Luzin-Souslin theorem in Anush's notes, because i is bijective, it's a Borel embedding (i.e. it maps Borel sets to Borel sets). This implies that $\mathcal{B}(\tau_1) \subseteq \mathcal{B}(\tau_0)$ as desired. ⋆

Problem 5. Prove the following characterization of Borel sets: A subset B of a Polish space X is Borel iff it is an injective continuous image of a closed subset of $\mathcal{N}$.

Solution. The forward direction follows from Corollary 11.20 in Anush's notes; namely, we may refine the Polish topology on X to make B clopen. Then, we may find a closed subset F of Baire space and a continous bijection $f \colon F \to B$, which would be continous wrt the original subspace topology on B because we first refined the topology on X. Then $f \colon F \to X$ is a continous injection with $f"F = B$. The backwards direction follows from the Luzin-Souslin theorem, as closed subsets of Baire space are Polish and such an injective continuous function would be a Borel embedding. ⋆

Problem 6. Let X be a Polish space.

1. Show that $\mathcal{F}(\omega^\omega)$ admits a Borel selector.
2. By Problem 28, there is a continuous open surjection $g : \omega^\omega \to X$. Prove that the map $f : \mathcal{F}(X) \to F(\omega^\omega)$ defined by $F \mapsto g^{-1}(F)$ is Borel.
3. Conclude that $\mathcal{F}(X)$ admits a Borel selector.

Solution.

1. Given a closed subset F of ω^ω, let $s(F)$ be the left most branch of F. Such a branch exists by recursion; i.e. we know that $F = [T]$ for some tree pruned T on ω. Construct $(s_n)_{n<\omega} \in T^\omega$ such that $\text{dom}(s_n) = n$ and $s_n \subseteq s_{n+1}$ by letting $s_{n+1} = s_n^\frown k$, where k is the least number such that $s_n^\frown k \subseteq y$ for some $y \in [T]$. Such a k always exists because our tree is pruned. Then $s = \bigcup_n s_n$ is the leftmost branch through T (and therefore of F).

   To check this map is Borel, given $t \in \omega^{<\omega}$, we have that $F \in s^{-1}(N_t) \Leftrightarrow s(F) \supseteq t$. Observe though that this happens iff $F \cap N_t \neq \varnothing$ and for any $s \in \omega^{<\omega}$ lexicographically to the left of t, $F \cap N_s = \varnothing$. But, then $F \in s^{-1}(N_t) \Leftrightarrow F \in [N_t] \cap \bigcap_s [N_s]^c$. The RHS is Borel by the definition of the Effros Borel space because the intersection is countable. So the selector is indeed a Borel selector.

2. Fix $t \in \omega^{<\omega}$. Observe that $g"N_t$ is open. Then we have $F \in f^{-1}[N_t] \Leftrightarrow g^{-1}(F) \in [N_t] \Leftrightarrow g^{-1}(F) \cap N_t \neq \varnothing \Leftrightarrow F \cap g"N_t \neq \varnothing \Leftrightarrow F \in [g"N_t]$. It follows that the map f is Borel.

3. The map $g \circ s \circ f \colon \mathcal{F}(X) \to X$ is Borel by the previous parts of this problem. Definition chasing yields that this is a selector, as desired.

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Problem 7. For a topological space X, show that $BP(X)$ admits envelopes.

Solution. For a set $A \subseteq X$, consider the set $B = U(A^c)^c \cup A = U(A^c)^c \cup (U(A^c) \cap A))$. $U(A^c)^c$ has the BP, and since $U(A^c) \Vdash A^c$, we have that $U(A^c) \setminus A^c = U(A^c) \cap A$ is meager (and therefore has the BP). So, $B \in BP(X)$. To show that B is an envelope for A, fix $C \subseteq B \setminus A = U(A^c)^c \cap A^c$ and assume that C has the BP. It's enough to show that C is meager. Let $U =^* C$. If U is empty, then C is meager and we win. Otherwise, if $U \neq \varnothing$, then $U \Vdash C$. Since $C \subseteq A^c$, we have that $U \Vdash A^c$, implying that $U \subseteq U(A^c)$. Further, since $C \subseteq U(A^c)^c$, we have $U \Vdash U(A^c)^c$. This implies that $U \cap U(A^c) = U$ is meager. Since $U =^* C$, we get that C is meager as well. $\star$

Problem 8. Prove directly (without using Wadge's theorem or lemma) that any countable dense $Q \subseteq 2^\omega$ is $\mathbf{\Sigma}^0_2$ -complete.

Solution. Q is $\mathbf{\Sigma}^0_2$ because singletons are closed and Q is countable. Now, assume that X is a zero-dimensional Polish space and $A \subseteq X$ be $\mathbf{\Sigma}^0_2$. By Thm 5.8 in Anush's notes we may assume that X is a closed subset of ω^ω. Then, we may write $A = \bigcup_n C_n$ where each C_n is a closed subset of ω^ω. This implies that $C_n = [T_n]$ for some pruned tree T_n on ω.

For the sake of sanity, we hope the reader is happy with just a (hopefully clear) description of the map $f\colon \omega^\omega \to 2^\omega$. First, for each $t \in \omega^{<\omega}$, fix a $q_t \in Q \cap N_t$ and $r_t \in Q^c \cap N_t$, where if $s \sqsubseteq t$, then $q_s = q_t$ and $r_s = r_t$. We can do this because Q is countable and dense. Also, fix $x \in \omega^\omega$. The main idea is that we check if $x \in [T_n]$ one n at a time, by successively checking if initial segments of x are in T_n. When there's an initial segment of x that's not in T_n, we stop checking if $x \in [T_n]$ and move to check if $x \in [T_{n+1}]$.

The value of $f(x)(n)$ will depend on what we've checked at the n^{th} stage of our computation. In particular, assume we are checking if $x|k \in T_m$ at stage n. If, indeed, $x|k \in T_m$, then we set $f(x)(n) = q_{f(x)|n}(n)$. If this is the case, during the $n+1$ stage in our computation we will check if $x|(k+1) \in T_m$. Otherwise, we set $f(x)(n) = r_{f(x)|n}(n)$. If this is the case, during the $n+1$ stage in our computation we will check if $x|k \in T_{m+1}$. In the first case, $f(x)$ looks more like an element of Q. In the second case, $f(x)$ looks more like an element of Q^c.

Now, notice that, indeed, $x \in A \Leftrightarrow f(x) \in Q$ because we required that $q_s = q_t$ and $r_s = r_t$ if $s \sqsubseteq t$. To check continuity, fix $x \in \omega^\omega$ and $n < \omega$. We must show that the first n digits of $f(x)$ depends on the first m digits of x for some $m < \omega$. But this is the case because the first n digits of $f(x)$ depends on at most the first n digits of x (depending on whether or not initial segments of x are in the various T_i). It follows that $A \leq_W Q$ and so Q is $\mathbf{\Sigma}^0_2$-complete as desired. ⋆

Problem 9. Show that the set of eventually zero binary sequences is $\mathbf{\Sigma}^0_2$ complete. Conclude that the set of binary sequences with infinitely many 0's is $\mathbf{\Pi}^0_2$ complete.

Solution. Let Q_2 be the eventually zero binary sequences and N_2 the binary sequences with infinitely many 0's. Regarding the second part of this problem, if Q_2 is $\mathbf{\Sigma}^0_2$ complete, then the same argument would show that the eventually one sequences is also $\mathbf{\Sigma}^0_2$ complete. This implies that the complement (i.e. N_2) is $\mathbf{\Pi}^0_2$ complete.

Now, Q_2 is countable because elements of Q_2 are determined by finite initial segments (because they're eventually zero). It's also not hard to see that Q_2 is dense. So by the previous problem, Q_2 is $\mathbf{\Sigma}_2^0$ complete.

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